The importance of friction in mountain wave drag enhancement by parametric resonance

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Stratified flow over a 2D mountain

Linear theory and numerical simulations

- Boussinesq approximation
- Linearization ($l_0 h_0 << 1$)
- Non-hydrostatic flow ($l_0 a \neq \infty$)
- Viscous processes represented by Rayleigh damping $\lambda^2$

Bell-shaped ridge

$$h = \frac{h_0}{1 + (x/a)^2}$$

Basic Scorer parameter: $l_0 = N/U$

Linear model

$$N^2 = \frac{g}{\theta_0} \frac{d\theta}{dz}$$
Linear model

\[ \hat{w}'' + \left( \frac{l^2}{1 - i\lambda / U_k} - k^2 \right) \hat{w} = 0 \]

Taylor-Goldstein equation

- Boundary condition at surface:
- Radiation or decay boundary condition aloft

\[ \hat{w}(z = 0) = iU_k \hat{h} \]

Used Scorer parameter profiles:

\[ l^2 = l_0^2 \left[ 1 + \varepsilon \cos(nz + \phi) \right] \]

\[ \varepsilon = 0.1 \]

Calculation of gravity wave drag:

\[ D = \int_{-\infty}^{+\infty} p(z = 0) \frac{\partial h}{\partial x} \, dx = 2\pi i \int_{-\infty}^{+\infty} k \hat{p}^* (z = 0) \hat{h} \, dk \]
Solution procedure

Expand solution in power series of $\varepsilon$:

$$\hat{w} = \hat{w}_0 + \varepsilon \hat{w}_1 + \varepsilon^2 \hat{w}_2 + \ldots$$

Drag is also power series of $\varepsilon$:

$$D = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots$$

It is sufficient to expand $D$ up to first order:

$$\frac{D}{D_0} = 1 + \varepsilon \frac{D_1}{D_0} = 1 + 2\varepsilon \left[ \int_0^{+\infty} k' e^{-2k'} m'_R \frac{(4m'^2_R + 4m'^2_I - n'^2) \cos \phi - 4n'm'_I \sin \phi}{(4m'^2_R - 4m'^2_I - n'^2)^2 + 64m'^2_R m'^2_I} dk' \right]$$

$$k' = ka$$

$$n' = n/l_0$$

$$m'_R = m_R/l_0$$

$$m'_I = m_I/l_0$$

Vertical wavenumber of internal gravity waves

Normalized drag $D/D_0$ is function of: $\varepsilon$, $n/l_0$, $\phi$, $l_0 a$, $\lambda a/U$
Numerical simulations

Use 2D nonlinear nonhydrostatic model called FLEX (Argaín et al. 2009, BLM)

- Finite difference method, with implicit time integration
- Terrain-following grid with 192 x 525 grid points
- Domain of 240 km in horizontal and 44 km in vertical
- Local grid refinement near surface with $\Delta z \sim 32$ m
- Sponges with Rayleigh damping at lateral and top boundaries
- Raymond & Kuo boundary condition at lateral boundaries
- 4th-order spatial filter applied every 3 time steps
- $\Delta t = 2$ s, integration time up to 69 h, typically 17 h.

- Anelastic approximation used
- Inviscid flow (no turbulence closure)
- Free-slip surface boundary condition
- $U=20$ m/s, $N=0.01$ s$^{-1}$, $a=4,10,20$ km, $h_0=10$ m
- $l_0a=2,5,10$, $l_0h_0=5\times10^{-3}$ (strongly linear flow)
Results - Effect of friction

\[ l_0a = 5 \quad \varepsilon = 0.1 \]

(a) \( \phi = 0 \)
(b) \( \phi = \pi/2 \)
(c) \( \phi = \pi \)
(d) \( \phi = 3\pi/2 \)
Non-hydrostatic effects (analytical) \[ \varepsilon = 0.1 \quad \lambda a / U = 2 \times 10^{-2} \]

(a) \[ \phi = 0 \]

(b) \[ \phi = \pi / 2 \]

(c) \[ \phi = \pi \]

(d) \[ \phi = 3 \pi / 2 \]
Non-hydrostatic effects (numerical) $\varepsilon = 0.1$

- \(\phi = 0\)
- \(\phi = \pi/2\)
- \(\phi = \pi\)
- \(\phi = 3\pi/2\)
Pressure perturbation (analytical) \( \varepsilon = 0.1 \quad n/l_0 = 2 \)

Normalized pressure perturbation, \( p/[\rho_0 U^2 l_0 h_0] \) \( \lambda a/U = 2 \times 10^{-2} \)

- Pressure perturbation more symmetric (low drag) when \( \phi = \pi/2 \), and more antisymmetric (high drag) when \( \phi = 3\pi/2 \)
- Effect becomes weaker as \( l_0a \) decreases
Pressure - comparison with numerical results

Normalized pressure perturbation, $p/[\rho_0 U^2 l_0 h_0]$ 

$\varepsilon = 0.1 \quad n/l_0 = 2 \quad l_0 a = 5$

- Relative magnitude of pressure perturbation correctly predicted
- Some details of the flow different – e.g. absence of downstream maximum in blue curve
Flow structure

Normalized vertical velocity perturbation field, $w/[(h_0/a)U]$

Non-resonant case: $\varepsilon = 0$  $l_0a = 5$

No resonance $\lambda a / U = 2 \times 10^{-2}$

Absorbing layer above $l_0z/\pi=4$
Flow structure

Normalized vertical velocity perturbation field, $w/[(h_0/a)U]$

High-drag state: $\varepsilon = 0.1$, $n/l_0 = 2$, $l_0a = 5$

$\lambda a / U = 2 \times 10^{-2}$

Absorbing layer above $l_0z/\pi = 4$
Flow structure

Normalized vertical velocity perturbation field, $w/[(h_0/a)U]$

Low-drag state:

- $\varepsilon = 0.1$
- $n/l_0 = 2$
- $l_0a = 5$

$\lambda a/U = 2 \times 10^{-2}$

Absorbing layer above $l_0z/\pi = 4$
Effect of turbulence closures

\[ \varepsilon = 0.1 \quad l_0 a = 5 \quad \phi = \frac{3\pi}{2} \]

(from Wells and Vosper 2010)

- Smagorinsky-type or K-\(\varepsilon\) turbulence closure
- M-O scaling in surface layer

- Large variation in drag behaviour depending on turbulence closure (and on numerical details in “inviscid” conditions)
Summary

• Mountain wave drag may be significantly enhanced when Scorer parameter oscillates with height
• Gravity wave drag behaviour is reproduced qualitatively in linear framework including nonhydrostatic effects and friction
• Substantial fractional drag enhancement results from resonance when $n/l_0 \approx 2$, even if $\varepsilon = 0.1$ - parametric resonance
• Friction has important impact on drag behaviour, generally moderating resonance
• However, when $\phi = \pi/2$ or $\phi = 3\pi/2$, drag maxima or minima are totally suppressed in inviscid conditions
• Important for numerical models, because both turbulence closures and numerical dissipation introduce ‘friction’
• Non-hydrostatic effects moderate resonance, because of wave dispersion

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